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Point interactions in one-dimensional quantum mechanics with coupled channels

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Abstract
We explore generalized point interactions in one-dimensional quantum mechanics with two coupled channels. They represent possible self-adjoint extensions of the nonrelativistic kinetic-energy operator. Assuming time-reversal invariance we find a family of self-adjoint extensions with seven parameters. This can be compared with the one-channel case in which the corresponding number of parameters is three.

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1. Introduction

Generalized point interactions (GPIs), which represent possible self-adjoint extensions (SAEs) of the nonrelativistic kinetic-energy (KE) operator $-(\hbar^2/2m)\nabla^2$, have been a subject of considerable interest in recent years. A number of papers have appeared on this subject. We will quote some of those papers in due course as they become relevant to the context of this paper. We confine ourselves to one space dimension. This paper is a sequel to an earlier paper [1] in which the GPIs of the one-channel case were examined. This time we examine what happens in the presence of two coupled channels.

We consider a situation such that a light particle is interacting with a point target that is fixed at the origin. The target has N energy levels and its interaction with the particle can cause transitions of the target from one level to another. Then we say that there are N channels. The target is in the ground state in channel 1, in the first excited state in channel 2, etc. The particle–target system is described by a wavefunction $\psi(x,t)$ with N components, $\psi_i(x,t)$ $(i = 1, 2, \ldots, N)$ where x is the coordinate of the particle and t is the time. The word 'channels' can be confusing. Even when $N = 1$, the particle can be in different partial-wave states. In one space dimension there can be two partial waves with even and odd parities. If the interaction potential is asymmetric as a function of x, these two partial waves are coupled and the problem can be regarded as that of two channels: see, e.g., [2]. Throughout this paper, however, we mean by channels those associated with the N levels of the target. We focus on the case of two channels, $N = 2$.

In the case of one channel, there is a four-parameter family of GPIs (SAEs) [1, 3–10]. One of the four parameters, which is related to time reversal invariance, is actually a trivial one and is physically uninteresting [9]. If we require time-reversal invariance we obtain a three-parameter family of GPIs in the one-channel case. In section 2 we show that, in the two-channel case, there is a nine-parameter family of GPIs. Two of the nine parameters are related to time reversal. One of the two is the same as the trivial one that appears in the one-channel case. Requiring time-reversal invariance we obtain a seven-parameter family of GPIs. In section 3 we examine the transmission–reflection problem in terms of the GPIs. The results are summarized and discussed in section 4.

2. Self-adjoint extensions of the kinetic-energy operator

An operator, say A, is defined by specifying its action on every vector in a space or its dense domain that is smaller than the entire space. The adjoint $A^\dagger$ of operator A is defined such that

$$\langle\phi|A\psi\rangle = \langle A^\dagger\phi|\psi\rangle \tag{1}$$

for all ψ and ϕ. Here ψ forms the domain of A and ϕ the domain of $A^\dagger$. If the action and domain of A are the same as those of $A^\dagger$, i.e., $A^\dagger = A$, then the operator A is said to be self-adjoint. In the one-dimensional case, A is self-adjoint if

$$\int_{-\infty}^{\infty} \phi^\dagger A\psi \,\mathrm{d}x - \int_{-\infty}^{\infty} (A\phi)^\dagger \psi \,\mathrm{d}x = 0 \tag{2}$$

holds for any pair of normalizable wavefunctions $\psi(x)$ and $\phi(x)$ in the same domain. It is understood that the wavefunctions also depend on t in general. The $\phi^\dagger$ is the Hermitian adjoint of ϕ. In the one-channel case $\phi^\dagger(x)$ is simply the complex conjugate $\phi^*(x)$. In the two-channel case, the wavefunction has two components and we distinguish $\phi^\dagger(x)$ and $\phi^*(x)$.

Let us summarize the situation of the one-channel case which we attempt to extend to the two-channel case. Consider the KE operator with a possible point interaction at the origin

$$A = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} \tag{3}$$

where m is the mass of the particle. Equation (2) can be reduced to

$$-\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} (\phi^\dagger\psi'' - \phi''^\dagger\psi)\,\mathrm{d}x = \frac{\hbar^2}{2m}[\phi^\dagger\psi' - \phi'^\dagger\psi]_{-0}^{+0} = 0 \tag{4}$$

where $\psi' = \mathrm{d}\psi/\mathrm{d}x$. It is understood that $\psi(x)$ and $\phi(x)$ are both twice differentiable except at $x = 0$ and that they both vanish as $|x| \to \infty$. The $\psi(x)$ and $\phi(x)$ and their derivatives are in general discontinuous at $x = 0$.

Consider the boundary condition at $x = 0$ which was introduced by Gesztesy and Kirsh [5] and was extensively discussed by Šeba [7, 8]

$$\begin{pmatrix}\psi'(+0)\\ \psi(+0)\end{pmatrix} = U\begin{pmatrix}\psi'(-0)\\ \psi(-0)\end{pmatrix} \qquad U = \mathrm{e}^{\mathrm{i}\theta}\begin{pmatrix}\alpha & \beta\\ \delta & \gamma\end{pmatrix} \tag{5}$$

where θ, α, β, γ and δ are all real constants. The α, etc, are subject to the condition

$$\alpha\gamma - \beta\delta = 1. \tag{6}$$

This boundary condition guarantees (4). It represents a GPI at the origin. There are four independent parameters including θ. As was pointed out in [9], however, θ is redundant. This is in the sense that, although the wavefunction depends on θ, observable quantities such as the transmission and reflection probabilities, the energy eigenvalue and the probability density of a bound state are all independent of θ.[4] In many-body problems, θ may have subtle implications in relation to the symmetry of the wavefunction [11], but we do not consider many-body problems in this paper. If we require time-reversal invariance of the GPI, U and hence $\mathrm{e}^{\mathrm{i}\theta}$ have to be real. In [1, 5, 7, 8], θ was set to $\mathrm{e}^{\mathrm{i}\theta} = -1$. If the interaction is invariant under space reflection $x \to -x$, the boundary condition has to be invariant under $\psi(\pm 0) \to \psi(\mp 0)$ and $\psi'(\pm 0) \to -\psi'(\mp 0)$. This holds if and only if $\mathrm{e}^{\mathrm{i}\theta}$ is real and $\alpha = \gamma$.

Let us mention two special cases. For the usual δ function potential $V(x) = g\delta(x)$ where g is a real constant, we obtain

$$U = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \tag{7}$$

i.e., with $\mathrm{e}^{\mathrm{i}\theta} = -1$, we obtain $\alpha = -1, \beta = -g, \gamma = -1$ and $\delta = 0$. The so-called δ' interaction, which should not be confused with $\delta'(x) = \mathrm{d}\delta(x)/\mathrm{d}x$, is defined in terms of

$$U = \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} \tag{8}$$

i.e. (again with $\mathrm{e}^{\mathrm{i}\theta} = -1$) $\alpha = -1, \beta = 0, \gamma = -1$ and $\delta = -h$. Here h is a real constant. These two interactions are both invariant under space reflection.

We now turn to the two-channel case. The wavefunctions have two components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{9}$$

The KE operator is

$$A = -\sigma_0 \frac{\hbar^2}{2m} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \qquad \sigma_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

In (4) it is understood that $\phi^\dagger \psi' = \phi_1^* \psi_1' + \phi_2^* \psi_2'$, etc. The two-component wavefunctions $\psi(x)$ and $\phi(x)$ are both twice differentiable except at $x = 0$.

Let us assume the boundary condition

$$\begin{pmatrix} \psi_1'(+0) \\ \psi_2'(+0) \\ \psi_1(+0) \\ \psi_2(+0) \end{pmatrix} = U \begin{pmatrix} \psi_1'(-0) \\ \psi_2'(-0) \\ \psi_1(-0) \\ \psi_2(-0) \end{pmatrix} \qquad U = \mathrm{e}^{\mathrm{i}\theta} \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}. \tag{11}$$

The same boundary condition applies to ϕ. The α, β, γ and δ are all 2×2 constant matrices and U a 4×4 matrix. For simplicity let us assume that α, β, γ and δ are all Hermitian. In the one-channel case we pointed out that the parameter θ is physically uninteresting [9]. We will see that the same situation holds in the two-channel case. Boundary condition (4) leads to

$$\begin{aligned}(\phi^\dagger \psi' - \phi'^\dagger \psi)_{+0} = [&\phi'^\dagger(\delta\alpha - \alpha\delta)\psi' + \phi^\dagger(\gamma\beta - \beta\gamma)\psi \\ &+ \phi^\dagger(\gamma\alpha - \beta\delta)\psi' - \phi'^\dagger(\alpha\gamma - \delta\beta)\psi]_{-0}.\end{aligned} \tag{12}$$

In order that (12) is reduced to (4) we choose α, etc, such that

$$[\delta, \alpha] = 0 \qquad [\beta, \gamma] = 0 \tag{13}$$

$$\alpha\gamma - \delta\beta = \sigma_0 \tag{14}$$

where $[\delta, \alpha] = \delta\alpha - \alpha\delta$, etc.

[4] One of the referees gave an interesting remark which is essentially the following: consider, for example, a GPI on a one-dimensional circle. Then parameter θ becomes relevant and describes the Aharonov–Bohm magnetic flux penetrating that circle. We, however, think that θ is again trivial in the following sense. It is natural to require that the wavefunction be single-valued on the circle in the absence of the magnetic field. This leads to $\mathrm{e}^{\mathrm{i}\theta} = 1$.

There are more constraints on α, etc, that follow from (4). Rewrite (11) as

$$U^{-1}\begin{pmatrix}\psi_1'(+0)\\ \psi_2'(+0)\\ \psi_1(+0)\\ \psi_2(+0)\end{pmatrix} = \begin{pmatrix}\psi_1'(-0)\\ \psi_2'(-0)\\ \psi_1(-0)\\ \psi_2(-0)\end{pmatrix} \qquad U^{-1} = \mathrm{e}^{-\mathrm{i}\theta}\begin{pmatrix}\gamma & -\beta\\ -\delta & \alpha\end{pmatrix}. \tag{15}$$

Express $(\phi^\dagger\psi' - \phi'^\dagger\psi)_{-0}$ in terms of the wavefunctions and their derivatives at $x = +0$. Then we obtain (12) in which the following substitutions are done: $\pm 0 \to \mp 0, \alpha \to \gamma, \gamma \to \alpha$, $\beta \to \beta$ and $\delta \to \delta$. This leads to the additional constraints

$$[\delta, \gamma] = 0 \qquad [\beta, \alpha] = 0 \tag{16}$$

$$\gamma\alpha - \delta\beta = \sigma_0. \tag{17}$$

From (14) and (17) we obtain

$$[\alpha, \gamma] = 0 \qquad [\beta, \delta] = 0. \tag{18}$$

Therefore α, etc, all commute with one another.

Let us examine how many real independent parameters are involved in the boundary condition. We write α as

$$\alpha = a_0\sigma_0 + \sum_{i=1}^{3} a_i\sigma_i = a_0\sigma_0 + \mathbf{a}\cdot\sigma \tag{19}$$

where σ_i are the 2×2 Pauli matrices and a_0 and a_i with $i = 1, 2, 3$ are all real constants. We also write β, γ and δ in the form of (19) with real coefficients b_0, b_i, c_0, c_i, d_0 and d_i, respectively. Thus we have altogether 16 parameters to begin with. The $\mathbf{a}$ can be regarded as a vector in a three-dimensional parameter space. Note that

$$\delta\alpha = (d_0a_0 + \mathbf{d}\cdot\mathbf{a})\sigma_0 + (d_0\mathbf{a} + a_0\mathbf{d} + \mathrm{i}\mathbf{d}\times\mathbf{a})\cdot\sigma \tag{20}$$

etc. It then follows from $[\delta, \alpha] = 0$ that $\mathbf{d}\times\mathbf{a} = 0$. That is, $\mathbf{d}$ and $\mathbf{a}$ are parallel to each other. (This includes the situation in which $\mathbf{d} = 0$ and/or $\mathbf{a} = 0$.) The commutativity of α, etc, means that the four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are all parallel. Among the 12 components contained in the four vectors, only six can be taken as independent. In addition we still have four parameters, a_0, etc. We have two more constraints. If we write $\mathbf{a}$ as $\mathbf{a} = a\mathbf{n}$ where $\mathbf{n}$ is a unit vector and similarly for the other three vectors with common $\mathbf{n}$, (14) leads to

$$a_0c_0 - b_0d_0 + ac - bd = 1 \tag{21}$$

$$a_0c + ac_0 - bd_0 - b_0d = 0. \tag{22}$$

Since $[\alpha, \gamma] = 0$, (17) is reduced to (14). Thus we obtain eight $(8 = 4 + 6 - 2)$ real independent parameters. Let us add that (13) and (14) guarantee $U^{-1}U = 1$ but not $UU^{-1} = 1$. The latter requires (16) and (17).

We now require that the system under consideration is invariant with respect to time reversal. The usual interpretation of time-reversal invariance is as follows. If wavefunction $\psi(x, t)$ is an admissible solution of the time-dependent Schrödinger equation, then its complex conjugate with time t replaced by $-t$, $\psi^*(x, -t)$, is also admissible. For a stationary state, apart from its time-dependent factor $\mathrm{e}^{-\mathrm{i}Et/\hbar}$, the wavefunction can be taken as a real function. If we take the complex conjugate of (11), U is replaced by U^*. The usual interpretation of time-reversal invariance requires that $U = U^*$. Recall however that $\sigma_2^* = -\sigma_2$ while σ_1 and σ_3 are real. Hence we find $\alpha(= a_0 + a\mathbf{n}\cdot\sigma) \neq \alpha^*$ unless $n_2 = 0$ and similarly $\beta \neq \beta^*$, etc. This contradicts $U = U^*$. Note, however, that $\mathbf{n}\cdot\sigma$ is invariant under rotation in the parameter space. Hence we can choose the coordinate axes (by rotating the axes around axis 3) such that the σ_2 term disappears. More explicitly let us rewrite $\psi(x, t)$ as

$$\psi(x, t) = \mathrm{e}^{(-\mathrm{i}/2)\theta_3\sigma_3}\chi(x, t) \qquad \tan\theta_3 = n_2/n_1. \tag{23}$$

In (11) replace ψ and ψ' with χ and χ', respectively. Then U is replaced by $\mathrm{e}^{(-\mathrm{i}/2)\theta_3\sigma_3}U\,\mathrm{e}^{(\mathrm{i}/2)\theta_3\sigma_3}$ in which $\mathbf{n}\cdot\sigma$ is replaced by

$$\mathrm{e}^{(\mathrm{i}/2)\theta_3\sigma_3}(\mathbf{n}\cdot\sigma)\,\mathrm{e}^{(-\mathrm{i}/2)\theta_3\sigma_3} = \left(n_1^2 + n_2^2\right)^{1/2}\sigma_1 + n_3\sigma_3. \tag{24}$$

The complex σ_2 has disappeared. The boundary condition for $\chi^*(x, -t)$ is exactly the same as that for $\chi(x, t)$. Before examining the time-reversal aspect we had eight parameters. When time-reversal invariance as interpreted above is imposed, we are left with seven physically interesting parameters.

Since parameter θ_3 can be eliminated as shown above, one may think that it is altogether redundant. This is, however, not necessarily the case. Suppose that the Hamiltonian of the system contains an interaction other than one of the SAEs of the kinetic-energy operator. For example, let the additional interaction be

$$V(x) = \sum_{i=1}^{3} V_i(x)\sigma_i \tag{25}$$

where $V_i(x)$ is an arbitrary real function of x with a finite range. After transformation (24) $V(x)$ becomes

$$(V_1\cos\theta_3 + V_2\sin\theta_3)\sigma_1 + (-V_1\sin\theta_3 + V_2\cos\theta_3)\sigma_2 + V_3\sigma_3. \tag{26}$$

The σ_2 term remains unless $V_2(x)/V_1(x) = n_2/n_1$. In this case θ_3 is a physically relevant parameter. The remaining interaction and hence observable quantities generally depend on θ_3. The σ_2 term of (26) violates time-reversal invariance[5]. We can make the system time-reversal invariant by choosing $\theta_3 = 0$, i.e., $n_2 = 0$ and $V_2(x) = 0$.

Finally let us examine the case in which boundary condition (11) is invariant under space reflection, i.e., $x \to -x$, $\psi(\pm 0) \to \psi(\mp 0)$ and $\psi'(\pm 0) \to -\psi'(\mp 0)$. When this reflection is performed, (11) is transformed to

$$\begin{pmatrix}\psi_1'(+0)\\ \psi_2'(+0)\\ \psi_1(+0)\\ \psi_2(+0)\end{pmatrix} = \mathrm{e}^{-\mathrm{i}\theta}\begin{pmatrix}\gamma & \beta\\ \delta & \alpha\end{pmatrix}\begin{pmatrix}\psi_1'(-0)\\ \psi_2'(-0)\\ \psi_1(-0)\\ \psi_2(-0)\end{pmatrix}. \tag{27}$$

The invariance with respect to space reflection requires that $\mathrm{e}^{\mathrm{i}\theta}$ is real and

$$\alpha = \gamma \tag{28}$$

namely, $a_0 = c_0$ and $a = c$. The number of independent parameters is reduced from seven to five. The corresponding number for the one-channel case is two.

Let us give two explicit examples. We start with the model with the two-channel interaction

$$V(x) = \frac{\hbar^2}{2m}\begin{pmatrix}g_1\delta(x) & g_{12}\delta(x)\\ g_{21}\delta(x) & g_2\delta(x)\end{pmatrix} \qquad g*_{12} = g_{21}. \tag{29}$$

It is understood that g_1 and g_2 are real but g_{12} and g_{21} can be complex provided that they are complex conjugates of each other. The $\delta(x)$ is the usual δ function. This interaction contains four independent parameters.

[5] If $V(x)$ represents an interaction between a spin and a magnetic field, then $V_i(x) \to -V_i(x)$ under the time-reversal operation. The σ_2 term does not violate time-reversal invariance. We are assuming, however, that $V_i(x)$ remains the same under the time-reversal operation.

By integrating the Schrödinger equation over the interval $(-\epsilon, \epsilon)$ and letting $\epsilon \to 0$, we determine the boundary condition on the wavefunction at $x = 0$. Matrix U (with $e^{i\theta} = -1$) for this model is given by

$$U = -\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & g_1 & g_{12} \\ 0 & 1 & g_{21} & g_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{30}$$

Note that, when g_{12} and g_{21} are complex, matrix β has a nonvanishing component with σ_2. As we pointed out already this component can be eliminated. We leave the U as such, however, so that we can see what happens when g_{12} and g_{21} are complex. In section 3 we will illustrate a situation such that the complex phase of g_{12}, which can be identified with $-\theta_3$, is redundant.

There is a similarity between (7) and (30). This similarity suggests that we can define a two-channel version of the δ' interaction as follows:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ h_1 & h_{12} & 1 & 0 \\ h_{21} & h_2 & 0 & 1 \end{pmatrix} \qquad h_{12}^* = h_{21}. \tag{31}$$

The h_1 and h_2 are real. This U contains four parameters including the imaginary part of h_{12}. Note that $\alpha = \gamma$ in the above two examples. The boundary conditions are both invariant under space reflection.

3. Transmission–reflection problem

In section 2 we found a seven-parameter family of GPIs which are invariant with respect to time reversal. A question that naturally arises here is: is seven the maximum possible number of the parameters of such GPIs? In this connection it would be useful to examine the transmission–reflection problem and the S matrix that is associated with it. When a wave representing a particle is incident on a potential, it is partially transmitted and partially reflected. This can be described in terms of transmission matrices T_L and T_R and reflection matrices R_L and R_R. Suffices L and R refer to the situations in which a wave is incident from the left and right, respectively. The T and R are 2×2 matrices.

Let us assume that there is no interaction, such as the $V(x)$ of (25), other than one of the GPIs at the origin. If the wave is incident in channel 1 from the left, the wavefunction can be written as

$$\psi_1 = \begin{cases} e^{ik_1x} + R_{L11}\, e^{-ik_1x} & \text{for} \quad x < 0 \\ T_{L11}\, e^{ik_1x} & \text{for} \quad x > 0 \end{cases} \tag{32}$$

$$\psi_2 = \begin{cases} R_{L21}\, e^{-ik_2x} & \text{for} \quad x < 0 \\ T_{L21}\, e^{ik_2x} & \text{for} \quad x > 0. \end{cases} \tag{33}$$

As we said in section 1, we are considering a situation in which a particle is interacting with a point target fixed at the origin. The k_1 and k_2 are related by $(\hbar k_1)^2/(2m) = (\hbar k_2)^2/(2m) + \Delta E$ where ΔE is the excitation energy of the target. It is understood that $k_i > 0$.

When applied to the wavefunction given above, (11) becomes

$$\begin{pmatrix} ik_1T_{11} \\ ik_2T_{21} \\ T_{11} \\ T_{21} \end{pmatrix} = U \begin{pmatrix} ik_1(1 - R_{11}) \\ -ik_2R_{21} \\ 1 + R_{11} \\ R_{21} \end{pmatrix}. \tag{34}$$

We have suppressed suffix L for brevity. Equation (34) determines T_{i1} and R_{i1} ($i = 1$ or 2). Equation (34) is equivalent to

$$U^{-1}\begin{pmatrix} \mathrm{i}k_1T_{11} \\ \mathrm{i}k_2T_{21} \\ T_{11} \\ T_{21} \end{pmatrix} = \begin{pmatrix} \mathrm{i}k_1(1-R_{11}) \\ -\mathrm{i}k_2R_{21} \\ 1+R_{11} \\ R_{21} \end{pmatrix}. \tag{35}$$

To determine T_{i1} it is convenient to use (35) and eliminate R_{i1}. To determine $R_{\mathrm{L}i1}$ it is convenient to use (34) and eliminate $T_{\mathrm{L}i1}$. By repeating the same procedure for the case with an incident wave in channel 2, we can determine T_L and R_L which are 2×2 matrices.

The T_R and R_R of the case in which the wave is incident from the right can be related to T_L and R_L through space reflection. That is, the former can be obtained from the latter by substitutions: $\theta \to -\theta, k_i \to -k_i, \alpha \to \gamma, \gamma \to \alpha, \beta \to -\beta$ and $\delta \to -\delta$. The results are as follows:

$$T_\mathrm{L} = 2\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}[-\beta + k\delta k + \mathrm{i}(k\alpha + \gamma k)]^{-1}k \tag{36}$$

$$T_\mathrm{R} = 2\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\theta}[-\beta + k\delta k + \mathrm{i}(\alpha k + k\gamma)]^{-1}k \tag{37}$$

$$R_\mathrm{L} = [-\beta + k\delta k + \mathrm{i}(\alpha k + k\gamma)]^{-1}[\beta + k\delta k + \mathrm{i}(\alpha k - k\gamma)] \tag{38}$$

$$R_\mathrm{R} = [-\beta + k\delta k + \mathrm{i}(\alpha k + \gamma k)]^{-1}[\beta + k\delta k - \mathrm{i}(k\alpha - \gamma k)] \tag{39}$$

where

$$k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \qquad k^{-1} = \begin{pmatrix} 1/k_1 & 0 \\ 0 & 1/k_2 \end{pmatrix}. \tag{40}$$

Note that k and k^{-1} do not commute with α, etc unless $k_1 = k_2$ or α, etc are all diagonal. If α, etc are all diagonal, the model becomes trivial in the sense that the two channels are decoupled. It can be shown that the inverse matrices that appear in (36)–(39) exist. If $k_1 = k_2$, all the matrices commute with each other and the results given above are reduced to those of the corresponding ones of the one-channel case, i.e., (6–9) of [9]. Note that the transmission and reflection probabilities are independent of θ. When the boundary condition (11) is invariant under space reflection, i.e., $\alpha = \gamma$, we obtain $R_\mathrm{L} = R_\mathrm{R}$.

Let us apply the above to the δ function interaction model of (29) and (30) (with $\mathrm{e}^{\mathrm{i}\theta} = -1$). This model with $\alpha = \gamma = \sigma_0$ has left–right symmetry. So we suppress suffices L and R. We obtain

$$T = 2\mathrm{i}(\beta + 2\mathrm{i}k)^{-1}k = \frac{-2\mathrm{i}}{\Delta}\begin{pmatrix} k_1(g_2 - 2\mathrm{i}k_2) & -g_{12}k_2 \\ -g_{21}k_1 & k_2(g_1 - 2\mathrm{i}k_1) \end{pmatrix} \tag{41}$$

$$R = -(\beta + 2\mathrm{i}k)^{-1}\beta \tag{42}$$

$$= \frac{-1}{\Delta}\begin{pmatrix} g_1(g_2 - 2\mathrm{i}k_2) - g_{12}g_{21} & -2\mathrm{i}\,g_{12}k_2 \\ -2\mathrm{i}g_{21}k_1 & g_2(g_1 - 2\mathrm{i}k_1) - g_{21}g_{12} \end{pmatrix} \tag{43}$$

where

$$\Delta = (g_1 - 2\mathrm{i}k_1)(g_2 - 2\mathrm{i}k_2) - g_{12}g_{21}. \tag{44}$$

The probability current is conserved. For example, when the wave is incident from channel 1 we obtain

$$k_1(|T_{11}|^2 + |R_{11}|^2) + k_2(|T_{21}|^2 + |R_{21}|^2) = k_1. \tag{45}$$

In $|T_{ij}|^2$ and $|R_{11}|^2$, g_{12} and g_{21} appear always in the form of $|g_{12}|^2 = |g_{21}|^2$. The conservation holds irrespectively of whether or not g_{12} and g_{21} are complex.

The interaction can support a bound state with the wavefunction of the form of

$$\psi_1(x) = \eta_1 \,\mathrm{e}^{-\kappa_1|x|} \qquad \psi_2(x) = \eta_2 \,\mathrm{e}^{-\kappa_2|x|}. \tag{46}$$

The κ_1 and κ_2 and then the energy eigenvalue of the bound state can be determined by

$$(g_1 + 2\kappa_1)(g_2 + 2\kappa_2) - g_{12}g_{21} = 0 \tag{47}$$

and the condition $(\hbar\kappa_1)^2/(2m) = (\hbar\kappa_2)^2/(2m) - \Delta E$. Recall that g_1, g_2 and $g_{12}g_{21}$ are all real. Equation (47) leads to a real energy of the bound state. The coefficients of the wavefunction of (46) are subject to

$$\frac{\eta_1}{\eta_2} = \frac{-g_{12}}{g_1 + 2\kappa_1} = \frac{g_1 + 2\kappa_1}{-g_{21}}. \tag{48}$$

If g_{12} and g_{21} are complex, then η_1/η_2 is complex and so is the wavefunction. The expectation values of physical quantities such as the probability density in the bound state, however, are all independent of the complex phase of g_{12}. We pointed out in section 2 that, by applying a rotation around axis 3 in the parameter space, we can make β real. Then g_{12} and g_{21} are both replaced by $|g_{12}|$ (see (24)). The physical quantities obtained in this representation are the same as those obtained with complex g_{12} and g_{21}.

We can define the S-matrix in the way it was done in [2]. Instead of ψ_i of (32) and (33) let us use

$$\phi_{i,\mathrm{in}} = A_i\theta(-x)\,\mathrm{e}^{\mathrm{i}k_i x} + B_i\theta(x)\,\mathrm{e}^{-\mathrm{i}k_i x} \tag{49}$$

$$\phi_{i,\mathrm{out}} = A_i'\theta(x)\,\mathrm{e}^{\mathrm{i}k_i x} + B_i'\theta(-x)\,\mathrm{e}^{-\mathrm{i}k_i x} \tag{50}$$

where $\theta(x) = 1$ (0) if $x > 0$ ($x < 0$). We define the S-matrix by

$$\begin{pmatrix} A_1' \\ A_2' \\ B_1' \\ B_2' \end{pmatrix} = S \begin{pmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{pmatrix}. \tag{51}$$

The S is a 4×4 matrix. (There are two partial waves in each of the two channels.) It is related to the T and R by

$$S = \begin{pmatrix} S_{aa} & S_{ab} \\ S_{ba} & S_{bb} \end{pmatrix} = \begin{pmatrix} T_L & R_R \\ R_L & T_R \end{pmatrix}. \tag{52}$$

If we assume time-reversal invariance, the S-matrix of a two-channel system can be expressed in terms of a 4×4 K matrix which is real and symmetric. The S-matrix in its general form (with an arbitrary interaction that is not restricted to point interactions) can have ten independent real parameters. For the GPIs that conform to time-reversal invariance we found a seven-parameter family. Recall that, in the one-channel case, the number (three) of parameters involved in GPIs is the same as that of the general S-matrix. The fact that the general S-matrix of the two-channel case can accommodate ten parameters may imply that the GPIs that we have obtained do not exhaust all possibilities.

4. Summary and discussion

This paper is an extension of [1] in which the GPIs or the SAEs of the KE operator were examined in the one-channel case in one space-dimension. In the presence of two coupled channels, we found a family of GPIs with nine parameters. Two of the nine parameters, θ of (11) and θ_3 of (23), are related to time-reversal invariance. We reiterated that the parameter θ

is physically uninteresting. If the Hamiltonian of the system has no interaction other than one of the GPIs, θ_3 is a redundant parameter. Even if $\theta_3 \neq 0$, the system conforms to time-reversal invariance. In a more general situation such that the Hamiltonian contains an additional interaction of an arbitrary form, θ_3 becomes relevant. If $\theta_3 \neq 0$, time-reversal invariance is violated. Observable quantities in general depend on θ_3.

By requiring time-reversal invariance and leaving θ and θ_3 out, we obtain a seven-parameter family of GPIs. On the other hand, the *S*-matrix in its general form that appears in the problem of transmission and reflection with an arbitrary interaction can have ten real independent parameters. This may imply that we have not exhausted all possible GPIs. We assumed that the matrices α, β, γ and δ of (11) are all Hermitian. That was for simplicity. As far as we know, our analysis is the first one in which GPIs as SAEs of the KE operator of the two-channel case are examined.

In their interesting paper, Wu and Yu recently proposed a model of quantum memory, an essential component of any quantum computer [12]. They constructed the model by means of a point interaction which couples two channels in one space dimension. They used the so-called pseudo-potential which contains three parameters. In the one-channel case, the *S*-matrix that they obtained can be reproduced by means of our one-channel GPIs with three parameters. They used the same pseudo-potential with three parameters in the two-channel case. The GPIs of the two-channel case with seven parameters that we have presented will give a framework with which one can construct more general or alternative models along the lines suggested by Wu and Yu.

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